

# ON NONUNIQUENESS OF SOLUTIONS OF HAMILTON-JACOBI-BELLMAN EQUATIONS

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**ABSTRACT.** An example of a nonunique solution of the Cauchy problem of Hamilton-Jacobi-Bellman (HJB) equation with surprisingly regular Hamiltonian is introduced. The proposed Hamiltonian  $H(t, x, p)$  fulfills the local Lipschitz continuity with respect to the triple of variables  $(t, x, p)$ , in particular, with respect to the state variable  $x$ . Moreover, the mentioned Hamiltonian is convex with respect to  $p$  and possesses linear growth in  $p$ , so it satisfies the classical assumptions. Given HJB equation with the Hamiltonian satisfying the above conditions, two distinct lower semicontinuous solutions with the same final conditions are given. Moreover, one of the solutions is the value function of Bolza Problem. The definition of lower semicontinuous solution was proposed by Frankowska [9] and Barron-Jensen [4]. The example that is proposed in the current paper allows to understand better the role of Lipschitz-type condition in the uniqueness of the Cauchy problem solution of HJB equation.

**Keywords:** Hamilton-Jacobi-Bellman equation, optimal control theory, nonsmooth analysis, viscosity solution.

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## 1. INTRODUCTION

The classical Cauchy problem for the Hamilton-Jacobi-Bellman equation is a partial differential equation with the final condition

$$\begin{aligned} \text{HJB} \quad & -U_t + H(t, x, -U_x) = 0 \quad \text{in } ]0, T[ \times \mathbb{R}^n, \\ & U(T, x) = g(x) \quad \text{in } \mathbb{R}^n. \end{aligned}$$

If the Hamiltonian  $H(t, x, p)$  is convex with respect to  $p$ , then there are relations between solutions of HJB and optimization problems involving a function dual to the Hamiltonian. This function, called the Lagrangian and denoted by  $L$ , is derived from  $H$  with the use of Legendre-Fenchel transform, namely:

$$(1.1) \quad L(t, x, v) = \sup_{p \in \mathbb{R}^n} \{ \langle v, p \rangle - H(t, x, p) \}.$$

Here  $\langle v, p \rangle$  denotes the inner product of  $v$  and  $p$ . It is possible for  $L$  to assume value  $+\infty$ . We consider the problem of Bolza in  $(t_0, x_0)$  to be

$$(\mathcal{P}_{t_0, x_0}) \quad \text{minimize} \quad \Gamma(x(\cdot)) := g(x(T)) + \int_{t_0}^T L(t, x(t), \dot{x}(t)) dt,$$

where  $\Gamma$  is minimized over  $\mathcal{A}([t_0, T], \mathbb{R}^n)$  (the space of all absolutely continuous functions  $x : [t_0, T] \rightarrow \mathbb{R}^n$  satisfying  $x(t_0) = x_0$ ). We say that Hamiltonian  $H(t, x, p)$  satisfies the *classical optimality conditions*, if it fulfills locally the Lipschitz continuity with respect to the system  $(t, x, p)$ . In addition to this,  $H(t, x, p)$  is convex with respect to  $p$  and increases linearly in  $p$ . The well-known result of Rockafellar [21] states that if the Hamiltonian  $H$  satisfies the classical optimality conditions and the function  $g$  is lower semicontinuous (lsc), then there exists at least one solution of  $(\mathcal{P}_{t_0, x_0})$ .

The set  $\text{dom } f = \{x : f(x) \neq \pm\infty\}$  is called the *effective domain* of  $f$ . An extended-real-valued function is called *proper* if it never takes the value  $-\infty$  and  $\text{dom } f \neq \emptyset$ . When  $H(t, x, \cdot)$  is proper, lsc and convex for each  $(t, x)$ , then  $L(t, x, \cdot)$  also has these properties.

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Furthermore, if this is the case, we can retrieve  $H$  from  $L$  by performing the Legendre-Fenchel transform for a second time:

$$(1.2) \quad H(t, x, p) = \sup_{v \in \mathbb{R}^n} \{ \langle p, v \rangle - L(t, x, v) \}.$$

Thus we have a one-to-one correspondence between Hamiltonians and Lagrangians in this convex case, and every equation like HJB can be related to a problem of the form  $(\mathcal{P}_{t_0, x_0})$ .

**Definition 1.1.** The  $V : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is defined using of  $\Gamma$  as follows:

$$(1.3) \quad V(t_0, x_0) = \begin{cases} \inf \{ \Gamma(x(\cdot)) : x(\cdot) \in \mathcal{A}([t_0, T], \mathbb{R}^n), x(t_0) = x_0 \} & \text{if } t_0 < T, \\ g(x_0) & \text{if } t_0 = T. \end{cases}$$

If the value function is differentiable, it is well-known that it satisfies HJB in the classical sense. However, in many situations the value function is not differentiable. Then the solution of the HJB equation must be defined in nonsmooth sense in such a way that under quite general assumptions on  $H$  and  $g$ ,  $V$  is the unique solution of HJB. Since we use the nonsmooth analysis we need a notion of a subgradient. For a vector  $v \in \mathbb{R}^n$  and a function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\pm\infty\}$ ,  $v$  is a subgradient of  $f$  at  $x \in \text{dom } f$ , written  $v \in \partial f(x)$ , if

$$(1.4) \quad \liminf_{y \rightarrow x, y \neq x} \frac{f(y) - f(x) - \langle v, y - x \rangle}{|y - x|} \geq 0.$$

In 1980 Crandall and Lions [7] introduced a nonsmooth *viscosity solutions*, and Crandall, Evans, and Lions gave a simplified approach in [6]. Viscosity solutions attracted a lot of attention, and over subsequent years a sizable literature developed from many authors that dealt with it. Among other issues, existence and uniqueness of solutions was considered. Usually in the viscosity theory we have assumptions about continuity of solutions (see [2, 3]).

In 1990 Baron-Jensen [4] and Frankowska [9] introduced extend viscosity solutions to semicontinuous functions for Hamiltonian that is convex in  $p$  and provided a uniqueness result. Frankowska [9] called these solutions *lower semicontinuous solutions*.

**Definition 1.2.** A function  $U : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is lower semicontinuous solution of HJB if it satisfies the following:

- (i)  $U$  is lower semicontinuous function and  $U(T, x) = g(x)$ ;
- (ii) For every  $(t, x) \in \text{dom } U$ , every  $(p_t, p_x) \in \partial U(t, x)$  the following holds:

$$(1.5) \quad \begin{cases} -p_t + H(t, x, -p_x) = 0 & \text{if } t \in ]0, T[, \\ -p_t + H(t, x, -p_x) \geq 0 & \text{if } t = 0, \\ -p_t + H(t, x, -p_x) \leq 0 & \text{if } t = T. \end{cases}$$

The main goal of the current paper is to present an example of two distinct lower semicontinuous solutions of HJB with surprisingly regular Hamiltonian. The proposed Hamiltonian  $H : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is, firstly, convex with respect to  $p$ , secondly, it increases linearly in  $p$ , i.e.  $|H(t, x, p)| \leq 2|p|$  for any  $t \in [0, T]$ ,  $x, p \in \mathbb{R}$ , thirdly, it satisfies locally the Lipschitz continuity, i.e.

$$(LLC) \quad \begin{aligned} & \forall r > 0 \quad \exists k > 0 \quad \forall t, s \in [0, T] \quad \forall x, y \in rB \quad \forall p, q \in rB \\ & |H(t, x, p) - H(s, y, q)| \leq k(|t - s| + |x - y| + |p - q|), \end{aligned}$$

where  $B$  is a closed unit ball. Thus the above Hamiltonian satisfies the classical optimality conditions. In addition, we show that one of the indicated lower semicontinuous functions is the value function given by (1.3). In many cases, the Lipschitz continuity (LLC) is sufficient for the solution of HJB to be unique. It turns out that, in general, for the uniqueness of

the solution, one needs some stronger Lipschitz-type condition, that we shall study further in connection to the results of uniqueness of solution of HJB.

Frankowska [9] proved that, the value function is the unique lower semicontinuous solution of the HJB equation if the Hamiltonian meets the classical optimality conditions and it is positively homogeneous in  $p$ , i.e.  $\forall_{r>0} H(t, x, rp) = rH(t, x, p)$ . Actually, the result of Frankowska does not require local Lipschitz continuity with respect to the triple  $(t, x, p)$ . It is enough to assume it is satisfied with respect to state variable  $x$  only. The example of nonuniqueness of solution of HJB introduced in the current paper, does not contradict the result of Frankowska as the Hamiltonian in our example, fulfills the classical optimality condition, but it is not positively homogeneous in  $p$ .

Earlier Ishii [13, Thm. 2.5] and Crandall-Lions [8, Thm. VI.1] had proved the uniqueness of viscosity solutions of HJB in the class of continuous functions. They had assumed instead of linear growth in  $p$  of Hamiltonian, the following condition

$$(1.6) \quad \begin{aligned} &\exists C > 0 \quad \forall t \in [0, T] \quad \forall x \in \mathbb{R}^n \quad \forall p, q \in \mathbb{R}^n \\ &|H(t, x, p) - H(t, x, q)| \leq C|p - q|. \end{aligned}$$

One can show that if the Hamiltonian is convex in  $p$  and possesses a linear growth of the form  $|H(t, x, p)| \leq 2|p|$ , then the condition (1.6) holds with constant  $C = 2$ . Therefore the Hamiltonian from our example of nonuniqueness also satisfies (1.6). Next, the results in [13, Thm. 2.5] and [8, Thm. VI.1] require the Lipschitz-type condition for the Hamiltonian with respect to state variable  $x$ ,

$$(SLC) \quad \begin{aligned} &\forall r > 0 \quad \exists k > 0 \quad \forall t, s \in [0, T] \quad \forall x, y \in rB \quad \forall p \in \mathbb{R}^n \\ &|H(t, x, p) - H(s, y, p)| \leq k(1 + |p|)(|t - s| + |x - y|), \end{aligned}$$

that is derived from the optimal control problem. The meaning of (SLC) in the optimal control problems is discussed in the papers of Rampazzo [20] and Frankowska-Sedrakyan [10]. The results in [13, Thm. 2.5] and [8, Thm. VI.1] do not require the convexity of the Hamiltonian in  $p$ , but the uniqueness of solution of HJB is obtained in the class of continuous functions. Using these results, Bardi and Capuzzo-Dolcetta [2, ch. V, Thm. 5.16] showed the uniqueness of solution of HJB in the class of lower semicontinuous functions assuming additionally the convexity of the Hamiltonian in  $p$ . Because the Hamiltonian in our example of nonuniqueness satisfies the classical optimality conditions, also the condition (1.6) is satisfied. It means that in the uniqueness results, the key point is the Lipschitz-type condition (SLC), at least in the case of lower semicontinuous solutions of the HJB equation.

The conditions presented above, needed for uniqueness of solution of the HJB equation, are classical conditions originated from optimal control theory. The conditions (1.6) and (SLC) in their stronger forms appear in the fundamental work of Crandall and Lions [7]. It follows from our example of nonuniqueness that the locally Lipschitz continuity (LLC) is not in general sufficient for the uniqueness of the solution of HJB. Therefore, the Lipschitz-type condition (SLC) has to be stronger than (LLC), because (SLC) implies the uniqueness of the solution of HJB. Let us now look more precisely at the difference between these two conditions. If Hamiltonian satisfies (LLC), putting  $p = q$  and increasing the right-hand side of the inequality, we get

$$(1.7) \quad \begin{aligned} |H(t, x, p) - H(s, y, p)| &\leq k(|t - s| + |x - y|) \\ &\leq k(1 + |p|)(|t - s| + |x - y|), \end{aligned}$$

for any  $t, s \in [0, T]$ ,  $x, y \in rB$  and  $p \in rB$ . We notice that the inequality (1.7) is identical with the one of (SLC). The only difference is that the inequality (1.7), assuming (LLC), holds only for  $p \in rB$ , and assuming (SLC), it holds for all  $p \in \mathbb{R}^n$ . This subtle difference

gives totally different results about the uniqueness of the solution of the HJB equation, as it is shown in our example of nonuniqueness. Finally, we notice that if the Hamiltonian is positively homogeneous in  $p$ , then the condition (SLC) follows from (LLC).

In order to understand better the reason for nonuniqueness of the solution of the HJB equation given in our example, we need to recall the Loewen-Rockafellar condition [15] that is more general version of (SLC). This condition is some kind of Aubin continuity of multifunction  $(t, x) \rightarrow \text{epi } L(t, x, \cdot)$ , where  $\text{epi } f(\cdot) = \{(x, r) : f(x) \leq r\}$ . This approach was used by Loewen-Rockafellar [15, 16] to study necessary conditions of optimal solution of Bolza problem  $(P_{t_0, x_0})$ . The results of this research used Galbraith from [11] in order to prove that the value function is the unique lower semicontinuous solution of HJB. Assuming the convexity of Hamiltonian in  $p$ , its mild growth in  $p$  and that it satisfies the Loewen-Rockafellar condition [15], the Galbraith uniqueness result of the solution of HJB has rather general nature including not only optimal control problems, but also variational problems. In our example of nonuniqueness, based on Galbraith result, we are able to indicate the element of construction of the Hamiltonian that corresponds to the lack of uniqueness. Namely, changing the corresponding element of construction, we obtain the Hamiltonian that does not satisfy the condition (SLC), while it satisfies Loewen-Rockafellar condition [15]. Therefore, by virtue of classical results, we are not able to say if after the mentioned change, we get the uniqueness of the solution of HJB or not. However, using the Galbraith result [11], we know that HJB with this Hamiltonian has the unique solution. Thus the changed element of the Hamiltonian construction must be responsible for the lack of uniqueness.

In the literature, the example of nonuniqueness of the solution of the equation of HJB is known. Crandall and Lions in their fundamental article [7] give an example of nonuniqueness of viscosity solution of the transport equation

$$(1.8) \quad U_t + b(x) \cdot U_x = 0.$$

The construction of their example is based on Beck results [5]. In this example, the Hamiltonian  $H(t, x, p) = b(x)p$  is continuous, but it does not satisfy the local Lipschitz continuity with respect to the state variable  $x$ . The example given by Crandall-Lions is natural, as one knows, that continuity is, in general, not sufficient condition for uniqueness of the solution of the Cauchy problem. Of course, if the function  $b(x)$  satisfies locally the Lipschitz continuity, then there exists the unique viscosity solution of the equation (1.8). Hence the continuity of  $b(x)$  is the necessary condition for finding two different viscosity solutions of (1.8). The Hamiltonian in our example of nonuniqueness is of different type from the one discussed above and satisfies locally the Lipschitz continuity with respect to the triple  $(t, x, p)$ , in particular, with respect to the state variable  $x$ . The solutions in the Crandall-Lions example are continuous, and in our example, they are lower semicontinuous. In connection to this, a question arises – do there exist two different continuous viscosity solutions of HJB with the Hamiltonian satisfying locally the Lipschitz continuity with respect to the state variable? We do not know the answer to this question but we believe that it is positive. Such an example of nonuniqueness would explain the meaning of Lipschitz-type condition in the results of uniqueness of the solution of HJB in the class of continuous functions.

Summarizing, our example of nonuniqueness shows that the key role in the uniqueness of viscosity solutions of HJB is played by the Lipschitz-type conditions originated in optimization instead of local Lipschitz continuity originated in differential equations theory. Therefore our result is presented in the aspect of optimization problems.

We discovered the example of nonuniqueness while we were working on generalization of results of Plaskacz-Quincampoix [19]. They are contained in [18].

## 2. EXISTENCE AND UNIQUENESS THEOREMS

In this section we present known theorems on existence and uniqueness of lower semicontinuous solutions of the HJB equation. We discuss results of the paper on the basis of these theorems. First, we introduce basic assumptions on Lagrangian.

- (L1)  $L : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is lower semicontinuous;
- (L2)  $L(t, x, v)$  is convex and proper with respect to  $v$  for every  $(t, x) \in [0, T] \times \mathbb{R}^n$ ;
- (L3)  $L(t, x, v) \geq 0$  for all  $(t, x, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ ;
- (L4) there exists a constant  $C > 0$  such that for every  $(t, x, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$  the following implication is satisfied  $|v| > C(1 + |x|) \Rightarrow L(t, x, v) = +\infty$ ;
- (L5) for any point  $(t, x, v)$  where  $L(t, x, v)$  is finite, and for any sequence  $(t_n, x_n) \rightarrow (t, x)$ , there exists a sequence  $v_n \rightarrow v$  along with  $L(t_n, x_n, v_n) \rightarrow L(t, x, v)$ .

Assumption (L4) is some kind of coercivity of the Lagrangian  $L$ , and the condition (L5) gives lower semicontinuity of the multifunction  $(t, x) \rightarrow \text{epi } L(t, x, \cdot)$  in the sense of Kuratowski. More information about (L5) can be found in the paper [22]. We say that a function  $\bar{x}(\cdot)$  at the point  $(t_0, x_0) \in \text{dom } V$ ,  $t_0 \in [0, T[$  is optimal trajectory of  $(\mathcal{P}_{t_0, x_0})$ , if it achieved the value  $V(t_0, x_0)$  in  $(\mathcal{P}_{t_0, x_0})$ . The function  $\bar{x}(\cdot)$  satisfying also Lipschitz continuity is called the *Lipschitz optimal trajectory*.

**Theorem 2.1 (Existence).** *Assume that  $L$  satisfies (L1)–(L5) and  $g$  is lsc and bounded from below. Let  $V$  be a value function associated with  $g$  and  $L$ . Then the value function  $V$  is bounded from below lower semicontinuous solution of the HJB equation with the Hamiltonian  $H$  given by (1.2). Moreover at every point  $(t_0, x_0) \in \text{dom } V$ ,  $t_0 \in [0, T[$  there exists the Lipschitz optimal trajectory of  $(\mathcal{P}_{t_0, x_0})$ .*

Theorem 2.1 is a consequence of more general Theorems 3.4 and 4.3 from [18]. Directly from the condition (L4) and Gronwall's inequality we have that optimal trajectories in Bolza problem  $(\mathcal{P}_{t_0, x_0})$  satisfy the Lipschitz continuity. In Section 3 we show that Lagrangian in our example on nonuniqueness satisfies conditions (L1)–(L5). Therefore, from Theorem 2.1 the value function is the lower semicontinuous solution of the HJB equation. Thus, the question is – if the value function is one of two distinct solutions of our example on nonuniqueness? The answer to this question is given in Section 5. Now we present an equivalent version of conditions (L1)–(L5) that are expressed by Hamiltonian.

**Proposition 2.2.** *Suppose that  $L$ ,  $H$  are related by (1.1), (1.2), respectively. Then  $L$  satisfies (L1)–(L5) if and only if  $L$  satisfies (H1)–(H4):*

- (H1)  $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  is upper semicontinuous;
- (H2)  $H(t, x, p)$  is convex with respect to  $p$  for every  $(t, x) \in [0, T] \times \mathbb{R}^n$ ;
- (H3) there exists a constant  $C > 0$  such that for all  $(t, x, p) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$  the following inequality is satisfied  $H(t, x, p) \leq C(1 + |x|)|p|$ ;
- (H4)  $H$  is lower semicontinuous on  $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ .

Condition (H3) is called a *linear growth in  $p$  of the Hamiltonian*. Proposition 2.2 come from properties of Legendre–Fenchel transform and Wijsman theorem, details can be found in [22, 23]. Having conditions (H1)–(H4) we can formulate the theorem on uniqueness of lower semicontinuous solution of the HJB equation.

**Theorem 2.3 (Uniqueness).** *Suppose that  $H$  satisfies (H1)–(H4) and (SLC). Let  $g$  be lsc and bounded from below, and  $V$  be a value function associated with  $g$  and  $L$ , where  $L$  is given by (1.1). If  $U$  is bounded from below lower semicontinuous solution of the HJB equation, then  $U = V$  on  $[0, T] \times \mathbb{R}^n$ .*

Theorem 2.3 is a particular case of Theorem 2.2 from [11]. In Section 3 we show that Hamiltonian from our example on nonuniqueness satisfies conditions (H1)–(H4) and the Lipschitz continuity (LLC). It means that the condition (SLC) is stronger than (LLC), since from Theorem 2.3 it implies uniqueness of solution of the HJB equation.

Now we discuss the Lipschitz-type condition Loewen-Rockafellar [15] that is a more general version of the condition (SLC). In Section 6 we show that this condition explains what is the reason of the lack of uniqueness in our example. We start with a subgradient characterization of the condition (SLC). A subgradient is defined for the function given on the whole Euclidean space (see Def. 1.4), so to use a subgradient to the Lagrangian  $L$  we extend  $L$  in the following way  $L(t, x, v) := L(0, x, v)$  for  $t < 0$  and  $L(t, x, v) := L(T, x, v)$  for  $t > T$ .

$$(2.1) \quad \begin{aligned} &\text{For every } r > 0 \text{ there exists } k > 0 \text{ such that at every point } (t, x, v) \\ &\in [0, T] \times rB \times \mathbb{R}^n, \text{ every } (w_1, w_2, p) \in \partial L(t, x, v) \text{ the inequality} \\ &|(w_1, w_2)| \leq k(1 + |p|) \text{ holds.} \end{aligned}$$

Condition (2.1) is equivalent to (SLC), if Hamiltonian satisfies (H1)–(H4). Moreover, one can prove that the condition (2.1) is a subgradient characterization of Lipschitz continuity of the multifunction  $(t, x) \rightarrow \text{epi } L(t, x, \cdot)$  in the Hausdorff's sense. Besides, it is easy to see that the condition (2.1) implies the following one:

$$(2.2) \quad \begin{aligned} &\text{For every } r > 0 \text{ there exists } k > 0 \text{ such that at every point } (t, x, v) \\ &\in [0, T] \times rB \times \mathbb{R}^n, \text{ every } (w_1, w_2, p) \in \partial L(t, x, v) \text{ we have} \\ &|(w_1, w_2)| \leq k(1 + |v| + |L(t, x, v)|)(1 + |p|). \end{aligned}$$

In Section 6 we show that the opposite implication to the above one is not true. It means that the condition (2.2) is a more general version of the condition (2.1). Furthermore, the condition (2.2) is a subgradient characterization of some kind of Aubin continuity of multifunction  $(t, x) \rightarrow \text{epi } L(t, x, \cdot)$ . This kind of Aubin continuity was introduced by Loewen-Rockafellar [14, Def. 2.3, (b)] to study optimal control problems with an unbounded differential inclusion. However, a subgradient characterization can be found in [12, Prop. 3.4]. The condition (2.2) in a slightly weaker subgradient version exists in papers of Loewen-Rockafellar [15, 16] where one investigates necessary conditions of optimal solutions of Bolza problem. It means that conditions (2.1) and (2.2) are strictly related to optimization problems. Using results of Loewen-Rockafellar [14, 15, 16] Galbraith proves in [11] that the value function is the unique lower semicontinuous solution of the HJB equation. He obtains this result assuming convexity of Hamiltonian with respect to  $p$ , a mild growth of Hamiltonian with respect to  $p$  and slightly more general Lipschitz-type condition than (2.2). Thus, if we replace the condition (SLC) by (2.2) in Theorem 2.3, then the claim of Theorem 2.3 still holds.

In [11, 12, 15, 16] one uses a limit subgradient that is more general than the regular subgradient that we use. We know that replacing the regular subgradient by the limit one the condition (2.2) is unchanged if we assume that Hamiltonian is continuous. Besides, without influencing the condition (2.2) we can replace  $(w_1, w_2, p) \in \partial L(t, x, v)$  by  $(w_1, w_2, v) \in \partial H(t, x, p)$  (see [11], Prop 2.6). We also know that the condition (2.2) implies (LLC) (see [11], Prop. 2.4). Therefore, assuming conditions (H1)–(H4) or (L1)–(L5) from the above argumentation we obtain

$$(2.3) \quad (\text{SLC}) \implies (2.2) \implies (\text{LLC}).$$

In the paper we show that the implications (2.3) cannot be reversed. Besides, we prove that the Lipschitz continuity (LLC) coming from differential equations is not sufficient for uniqueness of lower semicontinuous solution of the HJB equation. However, conditions (SLC) and (2.2) coming from optimization problems are sufficient for uniqueness.

### 3. EXAMPLE OF HAMILTONIAN

In this section we define Hamiltonian and discuss its regularity. In the next section we show that the HJB equality with this Hamiltonian does not have the unique lower semicontinuous solution. First, we define an auxiliary function  $\varphi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  by the formula

$$(3.1) \quad \varphi(t, x) = \sqrt{|t - x|} \exp\left(2\sqrt{|t - x|}\right).$$

The Hamiltonian  $H : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is defined by the formula

$$(3.2) \quad H(t, x, p) = \begin{cases} 0 & \text{if } 2|p| \leq \frac{1}{\varphi(t, x)}, t \neq x, \\ 2|p| - \frac{1}{\varphi(t, x)} & \text{if } 2|p| > \frac{1}{\varphi(t, x)}, t \neq x, \\ 0 & \text{if } p \in \mathbb{R}, t = x. \end{cases}$$

It is not difficult to see that the Hamiltonian  $H(t, x, p)$  given by (3.2) is continuous on  $[0, T] \times \mathbb{R} \times \mathbb{R}$ , convex with respect to  $p$  for each  $(t, x) \in [0, T] \times \mathbb{R}$  and has a linear growth in  $p$ , i.e.  $|H(t, x, p)| \leq 2|p|$  for all  $t \in [0, T]$ ,  $x, p \in \mathbb{R}$ , so it satisfies (H1)–(H4).

**Remark 3.1.** The function (3.1) does not satisfy the local Lipschitz continuity on the set  $[0, T] \times \mathbb{R}$ , because it is not Lipschitz on neighbourhoods of points  $(t, x)$  such that  $t = x$ . However, Hamiltonian (3.2) defined using the function (3.1) satisfies the local Lipschitz continuity on  $[0, T] \times \mathbb{R} \times \mathbb{R}$ .

Now we will prove that Hamiltonian (3.2) satisfies locally the Lipschitz continuity on  $[0, T] \times \mathbb{R} \times \mathbb{R}$ . We introduce some notations: by  $\text{LS}(\cdot)$  and  $\text{RS}(\cdot)$  we denote left and right side, respectively, of an equality or an inequality  $(\cdot)$ . Besides, by  $B(x, r)$  we denote a closed ball at a center  $x$  and radius  $r$ .

**Theorem 3.2.** *The Hamiltonian  $H$  given by (3.2) satisfies locally the Lipschitz continuity, i.e. for each  $(t_0, x_0, p_0) \in [0, T] \times \mathbb{R} \times \mathbb{R}$  there exist numbers  $r, k > 0$  such that*

$$(3.3) \quad \begin{aligned} &\forall t, s \in B(t_0, r) \quad \forall x, y \in B(x_0, r) \quad \forall p, q \in B(p_0, r) \\ &|H(t, x, p) - H(s, y, q)| \leq k(|t - s| + |x - y| + |p - q|). \end{aligned}$$

**Proof.** For  $\theta \geq 0$  we define an auxiliary function  $f$  by the formula  $f(\theta) = \sqrt{\theta} \exp(2\sqrt{\theta})$ . We notice that the function  $f$  is increasing and  $f(|t - x|) = \varphi(t, x)$ . Furthermore, the function  $f$  on  $[a, b]$  satisfies Lipschitz continuity if  $0 < a < b$ . We fix  $t_0 \in [0, T]$ ,  $x_0, p_0 \in \mathbb{R}$  and consider two cases

**Case 1.** Let  $p_0 \in \mathbb{R}$  and  $t_0 = x_0$ . We define  $r := 1/[2 \exp(4)(1 + 2|p_0|)^2]$  and  $k := 1$ . We notice that  $|p| \leq r + |p_0|$  for  $p \in B(p_0, r)$  and  $|t - x| \leq 2r$  for  $t \in B(t_0, r)$ ,  $x \in B(x_0, r)$ , besides  $r \leq 1/2$ . Therefore, for  $t \neq x$  we obtain

$$2|p| \leq 2(r + |p_0|) \leq 1 + 2|p_0| = \frac{1}{\sqrt{2r} \exp(2)} \leq \frac{1}{f(2r)} \leq \frac{1}{f(|t - x|)} = \frac{1}{\varphi(t, x)}.$$

By the definition of the Hamiltonian  $H(t, x, p) = 0$  if  $p \in \mathbb{R}$ ,  $t = x$  and  $2|p| \leq 1/\varphi(t, x)$ ,  $t \neq x$ . Therefore,  $H(t, x, p) = 0$  for each  $t \in B(t_0, r)$ ,  $x \in B(x_0, r)$ ,  $p \in B(p_0, r)$ , so we have the inequality (3.3).

**Case 2.** Let  $p_0 \in \mathbb{R}$  and  $t_0 \neq x_0$ . We define  $r$  by the formula  $r = |t_0 - x_0|/3$ , then  $r \leq |t - x| \leq 5r$  for each  $t \in B(t_0, r)$ ,  $x \in B(x_0, r)$ . Let  $l$  be the Lipschitz constant of the function  $f$  on  $[r, 5r]$ . We define  $k$  by  $k := 2 + l/f^2(r)$ .

Since  $t \neq x$  for each  $t \in B(t_0, r)$ ,  $x \in B(x_0, r)$ , then by the definition of the Hamiltonian (3.2) we obtain the following relations

(a) For  $2|p| \leq 1/\varphi(t, x)$  and  $2|q| \leq 1/\varphi(s, y)$  we have the inequality

$$\text{LS}(3.3) = 0 \leq \text{RS}(3.3).$$

(b) For  $2|p| \geq 1/\varphi(t, x)$  and  $2|q| \geq 1/\varphi(s, y)$  we have inequalities

$$\begin{aligned} \text{LS(3.3)} &\leq 2|p - q| + \frac{1}{f(|t - x|)f(|s - y|)} |f(|t - x|) - f(|s - y|)| \\ &\leq 2|p - q| + \frac{l}{f^2(r)} (|t - s| + |x - y|) \\ &\leq \text{RS(3.3)}. \end{aligned}$$

(c) For  $2|p| \geq 1/\varphi(t, x)$  and  $2|q| \leq 1/\varphi(s, y)$  we have inequalities

$$\begin{aligned} \text{LS(3.3)} &\leq 2|p| - \frac{1}{\varphi(t, x)} + \frac{1}{\varphi(s, y)} - 2|q| \\ &\leq 2|p - q| + \frac{1}{f(|t - x|)f(|s - y|)} |f(|t - x|) - f(|s - y|)| \\ &\leq \text{RS(3.3)}. \end{aligned}$$

The consequence of cases (a)-(c) is the inequality (3.3).  $\square$

**Remark 3.3.** The local Lipschitz continuity from Theorem 3.2 is equivalent to the Lipschitz continuity follows from compactness of closed and bounded sets in Euclidean spaces  $\mathbb{R}^n$ . Therefore, by Theorem 3.2 Hamiltonian (3.2) satisfies the Lipschitz continuity (LLC).

Now we prove that Hamiltonian (3.2) does not satisfy the condition (SLC). The proof of this fact shows how condition (SLC) is violated for large values of  $p$ . Moreover, from Theorem 3.2 we have that the condition (LLC) is weaker than the condition (SLC).

**Proposition 3.4.** *Hamiltonian (3.2) does not satisfy the condition (SLC).*

**Proof.** Assume that the condition (SLC) is satisfied. Then setting  $s_0 = y_0$  and  $p_n = 1/\varphi(t_n, x_n)$ , where  $t_n \neq x_n$ , we get

$$\begin{aligned} 1/\varphi(t_n, x_n) &= |H(t_n, x_n, p_n) - H(s_0, y_0, p_n)| \\ &\leq k(1 + 1/\varphi(t_n, x_n))(|t_n - s_0| + |x_n - y_0|). \end{aligned}$$

Multiplying the above inequality by  $\varphi(t_n, x_n)$  we obtain the inequality

$$1 \leq k(\varphi(t_n, x_n) + 1)(|t_n - s_0| + |x_n - y_0|),$$

that gives us the contradiction if  $(t_n, x_n) \rightarrow (s_0, y_0)$ .  $\square$

The Lagrangian  $L : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$  is defined by the formula

$$(3.4) \quad L(t, x, v) = \begin{cases} +\infty & \text{if } |v| > 2, t \neq x, \\ \frac{|v|}{2\varphi(t, x)} & \text{if } |v| \leq 2, t \neq x, \\ 0 & \text{if } v = 0, t = x, \\ +\infty & \text{if } v \neq 0, t = x. \end{cases}$$

It is not difficult to prove that the Lagrangian (3.4) satisfies conditions (L1)–(L5). We show that the Hamiltonian (3.2) is Legendre–Fenchel transform of the Lagrangian (3.4) with respect to the last variable, i.e. it satisfies the equality (1.2).

**Proposition 3.5.** *The Hamiltonian  $H$  and the Lagrangian  $L$  given by (3.2) and (3.4), correspondingly, satisfy the equality (1.2).*



**Proof.** To prove the proposition we consider two cases  $t \neq x$  and  $t = x$ .

**Case 1.** Let  $t \neq x$ , then by the definition of the Lagrangian (3.4) we have

$$(3.5) \quad H(t, x, p) = \sup_{|v| \leq 2} \left\{ p \cdot v - \frac{|v|}{2\varphi(t, x)} \right\}$$

$$(3.6) \quad \leq \sup_{|v| \leq 2} |v| \left[ |p| - \frac{1}{2\varphi(t, x)} \right].$$

By the inequality (3.6) we obtain the following bounds

$$(3.7) \quad H(t, x, p) \leq 2|p| - \frac{1}{\varphi(t, x)} \quad \text{if} \quad 2|p| \geq \frac{1}{\varphi(t, x)}$$

$$(3.8) \quad H(t, x, p) \leq 0 \quad \text{if} \quad 2|p| \leq \frac{1}{\varphi(t, x)}$$

We notice that in the case (3.7) we obtain the opposite inequality if we set  $v = \pm 2$  in the equality (3.5). In the case (3.8) we get the opposite inequality, if we set  $v = 0$  in the equality (3.5). Therefore, for  $t \neq x$  the considered Hamiltonian and the Lagrangian satisfy the equality (1.2).

**Case 2.** Let  $t = x$ , then from the definition of the Lagrangian (3.4) we have

$$H(t, x, p) = p \cdot 0 - 0 = 0,$$

that finishes the proof.  $\square$

Now we prove that Hamiltonian (3.2) does not satisfy the condition (2.2). The proof of this fact shows how condition (2.2) is violated for large values of gradients. By Theorem 3.2 it means that the condition (2.2) is stronger than the condition (LLC). It implies that the second implication in (2.3) cannot be reversed. In Section 6 we will see that large values of gradients not necessarily violate condition (2.2).

**Proposition 3.6.** *Lagrangian (3.4) does not satisfy the condition (2.2).*

**Proof.** Let the Lagrangian  $L$  be given by the formula (3.4). Then for  $t > x$ ,  $v \in ]0, 2[$  and  $(w_1, w_2, p) \in \partial L(t, x, v)$  we have

$$p = \frac{1}{2\varphi(t, x)}, \quad -w_1 = w_2 = \frac{v}{2\varphi(t, x)} \left[ \frac{1}{2(t-x)} + \frac{1}{\sqrt{(t-x)}} \right].$$

Therefore, the left and right sides of the inequality (2.2) are given by

$$\text{LS}(2.2) = \sqrt{2} w_2, \quad \text{RS}(2.2) = k \left( 1 + v + \frac{v}{2\varphi(t, x)} \right) \left( 1 + \frac{1}{2\varphi(t, x)} \right).$$

Let  $t_n - x_n \rightarrow 0+$  and  $v_n = 2\varphi(t_n, x_n)$ . If the inequality (2.2) is satisfied, then for large  $n \in \mathbb{N}$  we have

$$\sqrt{2} \left[ \frac{1}{2(t_n - x_n)} + \frac{1}{\sqrt{(t_n - x_n)}} \right] \leq 2k(1 + \varphi(t_n, x_n)) \left( 1 + \frac{1}{2\varphi(t_n, x_n)} \right).$$

Multiplying the above inequality by  $2(t_n - x_n)$  we have the inequality

$$\sqrt{2} + 2\sqrt{2(t_n - x_n)} \leq 2k [1 + \varphi(t_n, x_n)] \left( 2(t_n - x_n) + \frac{\sqrt{t_n - x_n}}{\exp(2\sqrt{t_n - x_n})} \right).$$

Taking it to the limit, we obtain contradiction.  $\square$

#### 4. SOLUTIONS OF HJB EQUATION

In this section we present two different, bounded, lower semicontinuous solutions of the HJB equation with the Hamiltonian  $H$  given by (3.2) and the cost function  $g : \mathbb{R} \rightarrow \mathbb{R}$  given by the formula

$$(4.1) \quad g(x) = \begin{cases} \exp(-2\sqrt{x-T}) - 1 & \text{if } x \geq T, \\ 1 & \text{if } x < T. \end{cases}$$

Now we prove the lemma that is needed in the proof of the inequality (1.5). For the function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $z \in \text{dom } f$  we define

$$(4.2) \quad \Delta f(z)(e) = \lim_{\tau \rightarrow 0+} \frac{f(z + \tau e) - f(z)}{\tau}.$$

**Lemma 4.1** ([1, Prop. 6.4.8]). *Let the function  $F : \mathbb{R}^2 \rightarrow \mathbb{R} \cup \{+\infty\}$  and a point  $(t, x) \in \text{dom } F$  be given. Assume that  $\Delta F(t, x)(v, \omega)$  exists. Then for each  $(p_t, p_x) \in \partial F(t, x)$  we have*

$$(4.3) \quad \Delta F(t, x)(v, \omega) \geq p_t v + p_x \omega.$$

In this paper we define some notions on the whole Euclidean space, that while applied to  $U(t, x)$  given on  $[0, T] \times \mathbb{R}^n$  we extend  $U(t, x)$  by setting  $+\infty$  for  $(t, x) \notin [0, T] \times \mathbb{R}^n$ . In addition to this, we introduce the following notation

$$\psi(t, x) = \frac{1}{\varphi(t, x)} \quad \text{for } t \neq x.$$

**4.1. First solution.** Let the function  $U : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  be given by the formula

$$(4.4) \quad U(t, x) = \begin{cases} \exp(-2\sqrt{x-t}) - 1 & \text{if } x \geq t, \\ 1 & \text{if } x < t. \end{cases}$$

**Theorem 4.2.** *The function  $U$  given by (4.4) is bounded lower semicontinuous solution of the HJB equation with the Hamiltonian (3.2) and the terminal condition (4.1).*

**Proof.** It is not difficult to notice that the function  $U$  is lower semicontinuous, bounded and  $U(T, x) = g(x)$ . We will prove that the function  $U$  satisfies conditions (1.5). To this end, we consider seven cases.

**Case 1.** Let  $x > t$  and  $t \in ]0, T[$ . Then the function  $U$  is differentiable at  $(t, x)$ . Therefore,  $\partial U(t, x) = \{(p_t, p_x)\}$ , where  $p_t = \psi(t, x) = -p_x$ . Since  $-2p_x \geq \psi(t, x)$ , then by the definition of  $H$  we have

$$\begin{aligned} -p_t + H(t, x, -p_x) &= -p_t - 2p_x - \psi(t, x) \\ &= -\psi(t, x) + 2\psi(t, x) - \psi(t, x) \\ &= 0. \end{aligned}$$

**Case 2.** Let  $x > t$  and  $t = 0$ . Then the equality  $\Delta U(t, x)(1, 0) = \psi(t, x)$  holds. Therefore, from the inequality (4.3) we obtain  $\psi(t, x) \geq p_t$  for each  $(p_t, p_x) \in \partial U(t, x)$ . Moreover,  $\Delta U(t, x)(0, 1) = -\psi(t, x)$  and  $\Delta U(t, x)(0, -1) = \psi(t, x)$ . Therefore, from the inequality (4.3) we have  $-\psi(t, x) \geq p_x$  and  $\psi(t, x) \geq -p_x$  for each  $(p_t, p_x) \in \partial U(t, x)$ . Since  $-p_t \geq -\psi(t, x)$  and  $-p_x = \psi(t, x)$  for each  $(p_t, p_x) \in \partial U(t, x)$ , then by the definition of  $H$  we have

$$\begin{aligned} -p_t + H(t, x, -p_x) &= -p_t - 2p_x - \psi(t, x) \\ &\geq -\psi(t, x) + 2\psi(t, x) - \psi(t, x) \\ &= 0. \end{aligned}$$

**Case 3.** Let  $x > t$  and  $t = T$ . Then the equality  $\Delta U(t, x)(-1, 0) = -\psi(t, x)$  holds. Therefore, from the inequality (4.3) we have  $-\psi(t, x) \geq -p_t$  for each  $(p_t, p_x) \in \partial U(t, x)$ . Moreover,  $\Delta U(t, x)(0, 1) = -\psi(t, x)$  and  $\Delta U(t, x)(0, -1) = \psi(t, x)$ . Therefore, from the inequality (4.3) we get  $-\psi(t, x) \geq p_x$  and  $\psi(t, x) \geq -p_x$  for each  $(p_t, p_x) \in \partial U(t, x)$ . Since  $-p_t \leq -\psi(t, x)$  and  $-p_x = \psi(t, x)$  for each  $(p_t, p_x) \in \partial U(t, x)$ , then by the definition of  $H$  we have

$$\begin{aligned} -p_t + H(t, x, -p_x) &= -p_t - 2p_x - \psi(t, x) \\ &\leq -\psi(t, x) + 2\psi(t, x) - \psi(t, x) \\ &= 0. \end{aligned}$$

**Case 4.** Let  $x = t$  and  $t \in [0, T]$ . Then the equality  $\Delta U(t, x)(0, 1) = -\infty$  holds. If  $(p_t, p_x) \in \partial U(t, x)$ , then from the inequality (4.3) we have the contradiction  $-\infty = \Delta U(t, x)(0, 1) \geq p_x \in \mathbb{R}$ . Therefore  $\partial U(t, x) = \emptyset$ .

**Case 5.** Let  $x < t$  and  $t \in ]0, T[$ . Then the function  $U$  is differentiable at  $(t, x)$ . Therefore,  $\partial U(t, x) = \{(p_t, p_x)\}$ , where  $p_t = 0 = p_x$ . By the definition of  $H$  we have

$$\begin{aligned} -p_t + H(t, x, -p_x) &= 0 + H(t, x, 0) \\ &= 0. \end{aligned}$$

**Case 6.** Let  $x < t$  and  $t = 0$ . Then the equality  $\Delta U(t, x)(1, 0) = 0$  holds. Therefore, from the inequality (4.3) we obtain  $0 \geq p_t$  for each  $(p_t, p_x) \in \partial U(t, x)$ . Moreover,  $\Delta U(t, x)(0, 1) = 0$  and  $\Delta U(t, x)(0, -1) = 0$ . Therefore, from the inequality (4.3) we have  $0 \geq p_x$  and  $0 \geq -p_x$  for each  $(p_t, p_x) \in \partial U(t, x)$ . Since  $-p_t \geq 0$  and  $-p_x = 0$  for each  $(p_t, p_x) \in \partial U(t, x)$ , then by the definition of  $H$  we have

$$\begin{aligned} -p_t + H(t, x, -p_x) &\geq 0 + H(t, x, 0) \\ &= 0. \end{aligned}$$

**Case 7.** Let  $x < t$  and  $t = T$ . Then the equality  $\Delta U(t, x)(-1, 0) = 0$  holds. Therefore, from the inequality (4.3) we have  $0 \geq -p_t$  for each  $(p_t, p_x) \in \partial U(t, x)$ . Moreover,  $\Delta U(t, x)(0, 1) = 0$  and  $\Delta U(t, x)(0, -1) = 0$ . Therefore, from the inequality (4.3) we get  $0 \geq p_x$  and  $0 \geq -p_x$  for each  $(p_t, p_x) \in \partial U(t, x)$ . Since  $-p_t \leq 0$  and  $-p_x = 0$  for each  $(p_t, p_x) \in \partial U(t, x)$ , then by the definition of  $H$  we have

$$\begin{aligned} -p_t + H(t, x, -p_x) &\leq 0 + H(t, x, 0) \\ &= 0, \end{aligned}$$

that finishes the proof.  $\square$

**4.2. Second solution.** Let the function  $V : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  be given by the formula

$$(4.5) \quad V(t, x) = \begin{cases} U(t, x) & \text{if } x \geq t, \\ 1 - \exp(-2\sqrt{t-x}) & \text{if } 2t - T \leq x < t, \\ 1 & \text{if } x < 2t - T. \end{cases}$$

**Theorem 4.3.** *The function  $V$  given by (4.5) is bounded lower semicontinuous solution of the HJB equation with the Hamiltonian (3.2) and the terminal condition (4.1).*

**Proof.** It is not difficult to notice that the function  $V$  is lower semicontinuous, bounded and  $V(T, x) = g(x)$ . We prove that the function  $V$  satisfies conditions (1.5). Since  $V(t, x) = U(t, x)$  for  $x \geq t$  and  $x < 2t - T$ , then by the Theorem 4.2 it is sufficient to show that  $V$  satisfies conditions (1.5), when  $2t - T \leq x < t$ . To do it we consider two cases.

**Case 1.** Let  $2t - T \leq x < t$  and  $t \in ]0, T[$ . Then  $\Delta V(t, x)(1, 2) = -\psi(t, x)$  and  $\Delta V(t, x)(-1, -2) = \psi(t, x)$ . Therefore, from the inequality (4.3) we have  $-\psi(t, x) \geq p_t + 2p_x$  and  $\psi(t, x) \geq -p_t - 2p_x$  for each  $(p_t, p_x) \in \partial V(t, x)$ . Moreover,  $\Delta V(t, x)(0, 1) = -\psi(t, x)$ , so by the inequality (4.3) we get  $-\psi(t, x) \geq p_x$  for each  $(p_t, p_x) \in \partial V(t, x)$ . Since  $-p_x \geq \psi(t, x)$  and  $\psi(t, x) = -p_t - 2p_x$  for all  $(p_t, p_x) \in \partial V(t, x)$ , then from the definition of  $H$  we have

$$\begin{aligned} -p_t + H(t, x, -p_x) &= -p_t - 2p_x - \psi(t, x) \\ &= \psi(t, x) - \psi(t, x) \\ &= 0, \end{aligned}$$

**Case 2.** Let  $2t - T \leq x < t$  and  $t = 0$ . Then  $\Delta V(t, x)(1, 2) = -\psi(t, x)$ . Therefore, from the inequality (4.3) we have  $-\psi(t, x) \geq p_t + 2p_x$  for all  $(p_t, p_x) \in \partial U(t, x)$ . Besides,  $\Delta V(t, x)(0, 1) = -\psi(t, x)$ , so from the inequality (4.3) we have  $-\psi(t, x) \geq p_x$  for all  $(p_t, p_x) \in \partial V(t, x)$ . Since  $-p_x \geq \psi(t, x)$  and  $-p_t - 2p_x \geq \psi(t, x)$  for all  $(p_t, p_x) \in \partial V(t, x)$ , then by the definition of  $H$  we have

$$\begin{aligned} -p_t + H(t, x, -p_x) &= -p_t - 2p_x - \psi(t, x) \\ &\geq \psi(t, x) - \psi(t, x) \\ &= 0, \end{aligned}$$

that ends the proof.  $\square$

## 5. THE VALUE FUNCTION

In this section we show that the function  $V$  given by (4.5) is the value function, in the sense of Definition 1.3, corresponding to the Lagrangian (3.4) and the cost function (4.1). It is not difficult to prove that a trajectory  $x : [t_0, T] \rightarrow \mathbb{R}$  given by  $x(t) = 2(t - t_0) + x_0$  for  $x_0 \geq 2t_0 - T$  and  $x(t) = x_0$  for  $x_0 < 2t_0 - T$  satisfy

$$(5.1) \quad V(t_0, x_0) = g(x(T)) + \int_{t_0}^T L(t, x(t), \dot{x}(t)) dt.$$

The problem to prove is on the basis of the Definition 1.1, that the trajectory  $x(\cdot)$  given above is optimal trajectory from Theorem 2.1. Therefore, to prove that the function (4.5) is the value function we use the paper [17] methods.

**5.1. Sketch of method.** Notation  $H_n \searrow H$  means that  $H_n \geq H_{n+1} \geq H$  for every  $n \in \mathbb{N}$  and  $H_n$  converges pointwise to  $H$ . Similarly  $L_n \nearrow L$  means that  $L_n \leq L_{n+1} \leq L$  for all  $n \in \mathbb{N}$  and  $L_n$  converges pointwise to  $L$ . Furthermore, we need the following results:

**Proposition 5.1** ([17, Cor. 3.2]). *Suppose that  $L_n, L$  and  $H_n, H$  are connected by relations (1.1) and (1.2). Then the following properties are equivalent:*

- (i) *The Hamiltonians  $H_n, H$  satisfy (H1)–(H3) and  $H_n \searrow H$ ,*
- (ii) *The Lagrangians  $L_n, L$  satisfy (L1)–(L4) and  $L_n \nearrow L$ .*

**Theorem 5.2** ([17, Thm. 3.3]). *Suppose that  $L_n, L$  satisfy (L1)–(L4) and  $L_n \nearrow L$ . Let  $g_n, g$  be bounded from below lower semicontinuous functions and  $g_n \nearrow g$ . If  $V_n, V$  are value functions corresponding to  $L_n, g_n$  and  $L, g$  respectively, then  $V_n \nearrow V$ .*

We construct a sequence of Hamiltonians  $(H_n)_{n \in \mathbb{N}}$  such that  $H_n$  satisfy (H1)–(H4) together with (SLC) and converge:  $H_n \searrow H$ . Then by Proposition 5.1 and Theorem 5.2 the value functions  $V_n$  corresponding to equations HJB with Hamiltonians  $H_n$  and cost functions  $g_n = g$  converge to the value function  $V$ . Since Hamiltonians  $H_n$  satisfy in addition

the condition (SLC), then the value functions  $V_n$  by Theorem 2.3 are unique lower semi-continuous solutions. Therefore, to find the value function  $V$ , one needs to find solutions of equations HJB with Hamiltonians  $H_n$  and the cost function  $g$  and then take the limit.

**Remark 5.3** ([17, Rem. 3.4]). Let the value functions  $V_n, V$  be as in Theorem 5.2. It can be proved that if functions  $x_n(\cdot)$  are optimal trajectories of  $V_n(t_0, x_0)$ , then accumulating points of the sequence  $(x_n(\cdot))_{n \in \mathbb{N}}$  are optimal trajectories of the function  $V(t_0, x_0)$ .

We use the above remark to find a solution to HJB equation.

**5.2. Approximation of the Hamiltonian.** Let the function  $\sigma : [0, +\infty[ \rightarrow \mathbb{R}$  be given as  $\sigma(z) = \sqrt{z}$ , and functions  $\sigma_n : [0, +\infty[ \rightarrow \mathbb{R}$  by

$$(5.2) \quad \sigma_n(z) = \begin{cases} \sqrt{z} & \text{if } z \geq \frac{1}{n}, \\ \frac{1}{\sqrt{n}} & \text{if } 0 \leq z < \frac{1}{n}. \end{cases}$$

We notice that functions  $\sigma_n$  satisfy locally the Lipschitz continuity and  $\sigma_n \searrow \sigma$ . Using functions  $\sigma_n$  we define functions  $\varphi_n : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$(5.3) \quad \varphi_n(t, x) = \sigma_n(|t - x|) \exp[2\sigma_n(|t - x|)].$$

Then functions  $\varphi_n$  also satisfy locally the Lipschitz continuity and  $\varphi_n \searrow \varphi$ .

Hamiltonians  $H_n : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  are defined by the formula

$$(5.4) \quad H_n(t, x, p) = \begin{cases} 0 & \text{if } 2|p| \leq \frac{1}{\varphi_n(t, x)}, \\ 2|p| - \frac{1}{\varphi_n(t, x)} & \text{if } 2|p| > \frac{1}{\varphi_n(t, x)}. \end{cases}$$

It is not difficult to notice that Hamiltonians  $H_n(t, x, p)$  given by (5.4) are continuous, convex with respect to  $p$  and have linear growth in  $p$ , so they satisfy conditions (H1)–(H4). Moreover  $H_n \searrow H$ , because  $\varphi_n \searrow \varphi$ . Similarly to the proof of Case 2 in Theorem 3.2 we can prove that Hamiltonians (5.4) satisfy the condition (SLC).

Lagrangians  $L_n : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$  are defined by the formula

$$(5.5) \quad L_n(t, x, v) = \begin{cases} \frac{|v|}{2\varphi_n(t, x)} & \text{if } |v| \leq 2, \\ +\infty & \text{if } |v| > 2. \end{cases}$$

It is easy to prove that Lagrangians (5.5) satisfy conditions (L1)–(L5) and  $L_n \nearrow L$ . Moreover, similarly like in the proof of Proposition 3.5 we can show that the Hamiltonian (5.4) is Legendre–Fenchel transform of the Lagrangian (5.5) with respect to the last variable.

**5.3. Value functions  $V_n$ .** We suppose that  $V$  given by (4.5) is the value function. Then by equalities (5.1) the optimal trajectory is given by formula  $\bar{x}(t) = 2(t - t_0) + x_0$  for  $x_0 \geq 2t_0 - T$  and  $\bar{x}(t) = x_0$  for  $x_0 < 2t_0 - T$ . Using Remark 5.3 we can expect that value functions  $V_n(t_0, x_0)$ , corresponding to Lagrangians (5.5) and a cost function (4.1), have optimal trajectories given by formulas  $\bar{x}_n(t) = 2(t - t_0) + x_0$  for  $x_0 \geq 2t_0 - T$  and  $\bar{x}_n(t) = x_0$  for  $x_0 < 2t_0 - T$ . If the above consideration is true then the following equality holds

$$V_n(t_0, x_0) = g(\bar{x}_n(T)) + \int_{t_0}^T L_n(t, \bar{x}_n(t), \dot{\bar{x}}_n(t)) dt.$$

Calculating in the above equality functions  $V_n : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  we get the formula:

$$V_n(t, x) =$$

For  $x \geq 2t - T$  we have

(a) if  $|t - x| \leq 1/n$  and  $T - 2t + x \geq 1/n$ , then

$$V_n(t, x) = (t - x)\sqrt{n} \exp(-2/\sqrt{n}) + (1 + 1/\sqrt{n}) \exp(-2/\sqrt{n}) - 1,$$

(b) if  $|t - x| \leq 1/n$  and  $T - 2t + x \leq 1/n$ , then

$$V_n(t, x) = \exp(-2\sqrt{T - 2t + x}) + (T - t)\sqrt{n} \exp(-2/\sqrt{n}) - 1,$$

(c) if  $t - x \geq 1/n$  and  $T - 2t + x \geq 1/n$ , then

$$V_n(t, x) = 2(1 + 1/\sqrt{n}) \exp(-2/\sqrt{n}) - \exp(-2\sqrt{t - x}) - 1,$$

(d) if  $t - x \geq 1/n$  and  $T - 2t + x \leq 1/n$ , then

$$\begin{aligned} V_n(t, x) &= [1 + \sqrt{n}(T - 2t + x) + 1/\sqrt{n}] \exp(-2/\sqrt{n}) \\ &\quad + \exp(-2\sqrt{T - 2t + x}) - \exp(-2\sqrt{t - x}) - 1, \end{aligned}$$

(e) if  $1/n \leq x - t$ , then  $V_n(t, x) = \exp(-2\sqrt{x - t}) - 1$ .

For  $x < 2t - T$  we have  $V_n(t, x) = 1$ .

**Remark 5.4.** In fact we do not know if the function  $V$  given by (4.5) is the value function. So we also do not know if functions  $V_n$  given by above formulas are value functions. The following results confirm that functions  $V_n$  and  $V$  are value functions.

**Proposition 5.5.** *Let Hamiltonians  $H_n$  be given by (5.4), the cost function  $g$  by (4.1), and the function  $V$  by (4.5). Then functions  $V_n$  given by the above formula are bounded lower semicontinuous solutions of equations HJB with Hamiltonians  $H_n$  and the cost function  $g$ , moreover  $V_n \rightarrow V$ .*

**Proof.** It is not difficult to notice that functions  $V_n$  are bounded (ie.  $-1 \leq V_n(\cdot, \cdot) \leq 1$ ) and  $V_n(T, x) = g(x)$ . We can prove that functions  $V_n$  are lower semicontinuous on  $[0, T] \times \mathbb{R}$ . Furthermore, functions  $V_n$  are differentiable on  $A = \{(t, x) \in ]0, T[ \times \mathbb{R} : x > 2t - T\}$  and  $B = \{(t, x) \in ]0, T[ \times \mathbb{R} : x < 2t - T\}$ , moreover  $\partial V(t, x) = \emptyset$  for  $x = 2t - T$  and  $t \in [0, T]$ , in addition  $V_n$  satisfy conditions (1.5) with Hamiltonians  $H_n$  and  $V_n \rightarrow V$ .  $\square$

**Remark 5.6.** Since functions  $V_n$  are lower semicontinuous solutions of equations HJB with Hamiltonians  $H_n$  and the cost function  $g$ , so by Theorem 2.3 the functions  $V_n$  are value functions. Therefore, from Theorem 5.2 the function  $V$  given by (4.5) is the value function, because  $V_n \rightarrow V$ .

## 6. REGULARITY OF HAMILTONIAN

By Remark 3.1 the function  $\varphi$  given by the formula (3.1) does not satisfy the local Lipschitz continuity. We show that it is the reason of the lack of the uniqueness solution of the HJB equation with Hamiltonian (3.2). Indeed, let us replace the function  $\varphi$  given by (3.1) by the function  $\widehat{\varphi}(t, x) = |t - x| \exp(2|t - x|)$ . Next define Hamiltonian  $\widehat{H}$  by the formula (3.2) using the function  $\widehat{\varphi}$ . Then Hamiltonian  $\widehat{H}$  satisfies conditions (H1)–(H4). Moreover, we show that using the local Lipschitz condition of the function  $\widehat{\varphi}$  Hamiltonian  $\widehat{H}$  satisfies the Loewen-Rockafellar condition (2.2). However, it does not satisfies the Lipschitz continuity (SLC) that easily follows from the proof of Proposition 3.4. Firstly, it means that we cannot reverse the first of implication (2.3). Secondly, by the result of Galbraith [11] we know

that the value function is the unique solution of the HJB equation with Hamiltonian  $\widehat{H}$ . Therefore, the reason of nonuniqueness of solutions of the HJB equation with Hamiltonian (3.2) is the lack of the local Lipschitz continuity of the function (3.1).

Let the function  $\varphi$  be given by (3.1) and  $w(\cdot, r)$  be modulus of continuity of the  $\varphi$  on the set  $[0, T] \times rB$ . Then the following proposition holds.

**Proposition 6.1.** *Let the Lagrangian  $L$  be given by (3.4) and the function  $\varphi$  be given by (3.1). Moreover, let  $w(\cdot, r)$  be modulus of continuity of  $\varphi$ . Then for every  $t, s \in [0, T]$  and  $x, y \in rB$ , every  $v \in \text{dom } L(t, x, \cdot)$  there exists  $\nu \in \text{dom } L(s, y, \cdot)$  such that*

- (i)  $|\nu - v| \leq 2(1 + |v| + |L(t, x, v)|) w(|s - t| + |y - x|, r);$
- (ii)  $L(s, y, \nu) \leq L(t, x, v) + 2(1 + |v| + |L(t, x, v)|) w(|s - t| + |y - x|, r).$

**Proof.** To prove the proposition we consider 3 cases.

**Case 1.** Let  $t \neq x$  and  $\varphi(s, y)/\varphi(t, x) \leq 1$ . Then for  $v \in \text{dom } L(t, x, \cdot)$  we put  $\nu = v \varphi(s, y)/\varphi(t, x)$ . Notice that  $\nu \in \text{dom } L(s, y, \cdot)$  and

$$\begin{aligned} \text{LS(i)} &= 2L(t, x, v)|\varphi(s, y) - \varphi(t, x)| \leq \text{RS(i)}, \\ \text{LS(ii)} &\leq L(t, x, v) \leq \text{RS(ii)}. \end{aligned}$$

**Case 2.** Let  $t \neq x$  and  $\varphi(s, y)/\varphi(t, x) > 1$ . Then for  $v \in \text{dom } L(t, x, \cdot)$  we put  $\nu = v$ . We notice that  $\nu \in \text{dom } L(s, y, \cdot)$  and

$$\begin{aligned} \text{LS(i)} &= 0 \leq \text{RS(i)}, \\ \text{LS(ii)} &\leq L(t, x, v) \leq \text{RS(ii)}. \end{aligned}$$

**Case 3.** Let  $t = x$ . If  $v \in \text{dom } L(t, x, \cdot)$ , then  $v = 0$ . Put  $\nu = 0$ . We notice that  $\nu \in \text{dom } L(s, y, \cdot)$  and

$$\begin{aligned} \text{LS(i)} &= 0 \leq \text{RS(i)}, \\ \text{LS(ii)} &= 0 \leq \text{RS(ii)}. \end{aligned}$$

Therefore, the proposition is proven.  $\square$

**Proposition 6.2.** *Suppose that the Lagrangian  $L$  satisfy (L1)–(L5). Then the condition (2.2) holds, if the following condition is true*

**(A)** *For every  $r > 0$  there exists  $k > 0$  such that for every  $t, s \in [0, T]$  and  $x, y \in rB$ , every  $v \in \text{dom } L(t, x, \cdot)$  there exists  $\nu \in \text{dom } L(s, y, \cdot)$  such that*

- (i)  $|\nu - v| \leq k(1 + |v| + |L(t, x, v)|)(|s - t| + |y - x|);$
- (ii)  $L(s, y, \nu) \leq L(t, x, v) + k(1 + |v| + |L(t, x, v)|)(|s - t| + |y - x|).$

**Proof.** We extend  $L$  in the following way:  $L(t, x, v) := L(0, x, v)$  for  $t < 0$  and  $L(t, x, v) := L(T, x, v)$  for  $t > T$ . Fix  $r > 0$  and choose  $k > 0$  for  $1 + r$  in such a way that the condition (A) holds. Let  $t \in [0, T]$ ,  $x \in rB$ ,  $v \in \mathbb{R}^n$  and  $(w_1, w_2, p) \in \partial L(t, x, v)$ . Without loss of generality we can assume that  $(w_1, w_2) \neq 0$ . Let  $(t_n, x_n) := (t, x) + (w_1, w_2)/[n|(w_1, w_2)|]$ , then  $x_n \in (1 + r)B$ . Since  $v \in \text{dom } L(t, x, \cdot)$ , then there exist  $v_n \in \text{dom } L(t_n, x_n, \cdot)$  such that

- (i)  $|v_n - v| \leq 2k(1 + |v| + |L(t, x, v)|) |(t_n, x_n) - (t, x)|,$
- (ii)  $L(t_n, x_n, v_n) \leq L(t, x, v) + 2k(1 + |v| + |L(t, x, v)|) |(t_n, x_n) - (t, x)|.$

We put  $b_n := n(v_n - v)$  and notice that (i) implies that  $|b_n| \leq 2k(1 + |v| + |L(t, x, v)|)$ . Therefore, a sequence  $\{b_n\}_{n \in \mathbb{N}}$  is bounded, so there exists a subsequence (denoted again by)  $b_n \rightarrow b$ . Obviously, the following inequality is satisfied

$$(6.1) \quad |b| \leq 2k(1 + |v| + |L(t, x, v)|)$$

Since  $(w_1, w_2, p) \in \partial L(t, x, v)$ , then the property [23, s. 301] and (ii) imply

$$\begin{aligned}
 \left\langle (w_1, w_2, p), \left( \frac{(w_1, w_2)}{|(w_1, w_2)|}, b \right) \right\rangle &\leq dL(t, x, v) \left( \frac{(w_1, w_2)}{|(w_1, w_2)|}, b \right) \\
 &\leq \liminf_n \frac{L(t_n, x_n, v_n) - L(t, x, v)}{|(t_n, x_n) - (t, x)|} \\
 (6.2) \qquad \qquad \qquad &\leq 2k(1 + |v| + |L(t, x, v)|).
 \end{aligned}$$

From the inequality 6.2 and 6.1 for  $(w_1, w_2, p) \in \partial L(t, x, v)$  we obtain

$$\begin{aligned}
 |(w_1, w_2)| &\leq 2k(1 + |v| + |L(t, x, v)|) + |p||b| \\
 &\leq 2k(1 + |v| + |L(t, x, v)|)(1 + |p|).
 \end{aligned}$$

So the proposition is proven.  $\square$

**Remark 6.3.** The only difference between the assertion of Proposition 6.1 and the condition (A) is contained in modulus. From Proposition 6.1 and 6.2 we obtain that if the function  $\varphi$  given by (3.1) is replaced by the function  $\widehat{\varphi}(t, x) = |t - x| \exp(2|t - x|)$ , then the Hamiltonian  $\widehat{H}$  given by (3.2) with  $\widehat{\varphi}$  satisfies the Lipschitz-type condition (2.2).

Crandall and Lions [7] constructed two continuous solutions of the transport equation (1.8). This construction is based on Beck results in [5]. We know that in the example of nonuniqueness of Crandall and Lions Hamiltonian  $H(t, x, p) = b(x)p$  cannot satisfy locally the Lipschitz continuity. In our example of nonuniqueness Hamiltonian satisfy locally the Lipschitz continuity. However, solutions are lower semicontinuous functions. In connection to this, there appears a question – if there exist two continuous solutions of HJB with the Hamiltonian satisfying locally the Lipschitz continuity? On the basis of results from this section we think that the answer to this question is positive. Indeed, we have proved that regularity of the function  $\varphi$  is responsible for the lack of uniqueness in our example. Similarly, in the example of Crandall and Lions the function  $b$  is responsible for the lack of uniqueness. Besides, we know that in the construction of two continuous solutions in the example of Crandall and Lions the significant role is played by Beck functions  $f, l$  that satisfy the equality  $b(x) := f'(f^{-1}(x)) = l'(l^{-1}x)$ . In fact, we are able to construct two lower semicontinuous solutions of transport equation using the method of Crandall and Lions without Beck result. To do it we need significantly less complex function  $b$ . We think that replacing in our example the function  $\varphi$  by more subtle Beck function  $b$  we can construct, using functions  $f$  and  $l$ , two continuous solutions of HJB equation with Hamiltonian satisfying locally the Lipschitz continuity. Notice that the example of Crandall and Lions, as opposed to our example, does not consider the meaning of Lipschitz type condition in uniqueness results. In particular, it does not consider that the local Lipschitz continuity is not sufficient to obtain uniqueness. We recall that in the classical theory of differential equations the local Lipschitz continuity is a natural assumption guaranteeing uniqueness of the Cauchy problem.

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